# Fast Corotated Elastic SPH Solids with Implicit Zero-Energy Mode Control

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## 1 KERNEL GRADIENT CORRECTION

Computing the deformation gradient using the standard SPH gradient in Eq. (6) leads to artifacts since it is not 1st-order consistent. To solve this problem we analyze the error of Eq. (6) by replacing $A(\mathbf{X}_j)$ with its Taylor approximation at the point $\mathbf{X}_i$

$$\begin{aligned}\sum_{j\in\mathcal{N}_i^0} V_j A_j \nabla W_{ij} &= \sum_{j\in\mathcal{N}_i^0} V_j \left(A_i + \nabla A_i \cdot \mathbf{X}_{ji} + O(\mathbf{X}_{ji}^2)\right)\nabla W_{ij}\\ &= \sum_{j\in\mathcal{N}_i^0} V_j A_i \nabla W_{ij} + \sum_{j\in\mathcal{N}_i^0}\left(V_j \nabla W_{ij}\otimes \mathbf{X}_{ji}\right)\nabla A_i + O(\mathbf{X}_{ji}^2),\end{aligned}$$

where $\mathbf{X}_{ji} = \mathbf{X}_j - \mathbf{X}_i$ and $\otimes$ denotes the dyadic product of two vectors $\mathbf{a}\otimes\mathbf{b} = \mathbf{a}\mathbf{b}^T$. To solve for $\nabla A$ we subtract the first and third term and multiply with the inverse of the matrix from the second term. This yields a 1st-order consistent approximation of the gradient

$$\nabla A_i \approx \sum_{j\in\mathcal{N}_i^0} V_j (A_j - A_i)\mathbf{L}_i \nabla W_{ij} \tag{1}$$

with the correction matrix [Bonet and Lok 1999]

$$\mathbf{L}_i = \left(\sum_{j\in\mathcal{N}_i^0} V_j \nabla W_{ij}\otimes \mathbf{X}_{ji}\right)^{-1}. \tag{2}$$

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### 1.1 Computation of Matrix $\mathrm{D}_i$

The matrix $\mathbf{D}_i \in \mathbb{R}^{9\times 3n}$, which was introduced in Eq. (12) to compute the deformation gradient, is a block matrix which is typically sparse. It has a $9 \times 3$ block for the particle $i$ starting at column $3i$:

$$(\mathbf{D}_i)_{0,3i} = -\sum_{j\in\mathcal{N}_i^0} V_j \begin{pmatrix} (\mathbf{L}_i \nabla W_{ij})_1 \mathbb{1} \\ (\mathbf{L}_i \nabla W_{ij})_2 \mathbb{1} \\ (\mathbf{L}_i \nabla W_{ij})_3 \mathbb{1} \end{pmatrix} \tag{3}$$

and one $9 \times 3$ block for each rest-pose neighbor particle $j$ starting at column $3j$:

$$(\mathbf{D}_i)_{0,3j} = V_j \begin{pmatrix} (\mathbf{L}_i \nabla W_{ij})_1 \mathbb{1} \\ (\mathbf{L}_i \nabla W_{ij})_2 \mathbb{1} \\ (\mathbf{L}_i \nabla W_{ij})_3 \mathbb{1} \end{pmatrix}. \tag{4}$$

Note that all components of the matrix $\mathbf{D}_i$ only depend on quantities from the rest pose which means that they are constant during the simulation.

### 1.2 Computation of Matrix $\mathrm{H}_{ij}$

In Eq. (21) the matrix $\mathbf{H}_{ij} \in \mathbb{R}^{3\times 3n}$ was introduced to compute the error vectors $\mathcal{E}_{ij}^i$. $\mathbf{H}_{ij}$ is also a block matrix and typically sparse. It has the a $3 \times 3$ block for particle $i$ starting at column $3i$:

$$(\mathbf{H}_{ij})_{0,3i} = -\sum_{k\in\mathcal{N}_i^0} V_k \, (\mathbf{L}_i \nabla W_{ik})^T \, \mathbf{X}_{ij} \mathbb{1} - \mathbb{1}. \tag{5}$$

For the rest-pose neighbor particle $j = k$ we get

$$(\mathbf{H}_{ij})_{0,3j} = V_k \left(\mathbf{L}_i \nabla W_{ij}\right)^T \mathbf{X}_{ij} \mathbb{1} + \mathbb{1}, \tag{6}$$

and finally for the neighbor particle $k \neq j$ we get

$$(\mathbf{H}_{ij})_{0,3k} = V_k \, (\mathbf{L}_i \nabla W_{ik})^T \, \mathbf{X}_{ij} \mathbb{1}. \tag{7}$$

Note that the matrix $\mathbf{H}_{ij}$ is constant during the simulation since its components only depend on quantities from the rest pose.

## REFERENCES

J. Bonet and T.-S. L. Lok. 1999. Variational and momentum preservation aspects of Smooth Particle Hydrodynamic formulations. *Computer Methods in Applied Mechanics and Engineering* 180, 1 (1999), 97 – 115.